

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

On existence, optimality and asymptotic stability of the Kalman filter with partially observed inputs[☆]Jinya Su^a, Baibing Li^{b,1}, Wen-Hua Chen^a^a Department of Aeronautical and Automotive Engineering, Loughborough University, Loughborough LE11 3TU, UK^b School of Business and Economics, Loughborough University, Loughborough LE11 3TU, UK

ARTICLE INFO

Article history:

Received 7 May 2014

Received in revised form

17 September 2014

Accepted 10 December 2014

Available online 12 January 2015

Keywords:

Asymptotic stability

Existence

Kalman filter

Optimality

Unknown inputs

ABSTRACT

For linear stochastic time-varying systems, we investigate the properties of the Kalman filter with partially observed inputs. We first establish the existence condition of a general linear filter when the unknown inputs are partially observed. Then we examine the optimality of the Kalman filter with partially observed inputs. Finally, on the basis of the established existence condition and optimality result, we investigate asymptotic stability of the filter for the corresponding time-invariant systems. It is shown that the results on existence and asymptotic stability obtained in this paper provide a unified approach to accommodating a variety of filtering scenarios as its special cases, including the classical Kalman filter and state estimation with unknown inputs.

© 2014 The Authors. Published by Elsevier Ltd.
This is an open access article under the CC BY license
(<http://creativecommons.org/licenses/by/4.0/>).

1. Introduction

State estimation plays an important role in state space modelling and control. It has been applied to a wide range of areas; see [Li \(2009\)](#) and [Liang, Chen, and Pan \(2010\)](#) for some recent applications in network control systems, transportation management, etc.

In the recent decades, state estimation for discrete-time linear stochastic systems with unknown inputs (also termed as unknown input filtering (UIF) problem) has received considerable attention since the original work of [Kitanidis \(1987\)](#) first appeared. Various filters were developed under different assumptions for the systems with unknown inputs; see, e.g., [Cheng, Ye, Wang, and Zhou \(2009\)](#), [Darouach and Zasadzinski \(1997\)](#), [Darouach, Zasadzinski, and Xu \(1994\)](#), [Fang and Callafon \(2012\)](#), [Gillijns and De Moor \(2007\)](#), [Hsieh \(2000, 2010\)](#) and [Kitanidis \(1987\)](#), among many others. Most of these researches used the technique of minimum variance unbiased estimation, hence leading to an unbiased minimum-variance

filter (UMVF). Another important research line is state estimation for descriptor systems ([Hsieh, 2011, 2013](#)). It has been recently shown in [Hsieh \(2013\)](#) that any linear descriptor systems can be transformed into a linear stochastic system with unknown inputs. This shows a close link between these two kinds of problem. In addition, various properties for these developed filters have been investigated, including the existence condition ([Darouach & Zasadzinski, 1997](#)), asymptotic stability ([Fang & Callafon, 2012](#)), and global optimality of the UMVF ([Cheng et al., 2009](#) and [Hsieh, 2010](#)).

Recently, [Li \(2013\)](#) has developed a Kalman filter for linear systems with partially observed inputs, where the inputs are observed not at the level of interest but rather the input information is available at an aggregate level. It has been shown that the developed filter provides a unified approach to state estimation for linear systems with Gaussian noise. In particular, it includes two important extreme scenarios as its special cases: (a) the filter where all the inputs are completely available (i.e. the classical Kalman filter; see, e.g., [Simon, 2006](#)); and (b) the filter where all inputs are unknown (i.e. the filter investigated in [Kitanidis, 1987](#) and many others for the UIF problem). Potentially the proposed filter can be applied to a variety of practical problems in many different areas such as population estimation and traffic control.

So far there is not any study discussing the existence and asymptotic stability issues of this newly proposed unified filter. In this paper we investigate the properties of the Kalman filter with partially observed inputs developed in [Li \(2013\)](#). For linear stochastic time-varying systems with partially observed inputs, we

[☆] This work was jointly funded by UK Engineering and Physical Sciences Research Council (EPSRC) and BAE Systems (EP/H501401/1). The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Tongwen Chen under the direction of Editor Ian R. Petersen.

E-mail addresses: J.Su2@lboro.ac.uk (J. Su), B.Li2@lboro.ac.uk (B. Li), w.chen@lboro.ac.uk (W.-H. Chen).

¹ Tel.: +44 1509 228841; fax: +44 1509 223960.

establish the existence condition for a general linear filter. Then we show that the developed filter is optimal in the sense of minimum error covariance matrix. Finally, we consider asymptotic stability of the filter for the corresponding time-invariant systems based on the established existence condition and optimality result.

This paper has provided a unified approach to accommodating existence and asymptotic stability conditions in a variety of filtering scenarios: it includes the results on existence and asymptotic stability for some important filters as its special cases, e.g., the filters developed for the problems where the inputs are completely available and where all the inputs are unknown. Note that the former is the classical Kalman filtering problem and the corresponding existence and asymptotic stability conditions are well established in the literature. For the latter case with unknown inputs, there has been a continuing research interest in existence and asymptotic stability conditions for various discrete-time systems (e.g., Cheng et al., 2009, Darouach & Zasadzinski, 1997, Fang & Callafon, 2012, Kitanidis, 1987) and continuous-time systems (e.g., Bejarano, Floquet, Perruquetti, & Zheng, 2013, Corless & Tu, 1998, Hou & Muller, 1992).

This paper is structured as follows. First, Section 2 is devoted to problem statement. Then we establish the existence condition in Section 3. We focus on the properties of the filter proposed in Li (2013) in Section 4. In Section 5, we investigate asymptotic stability. Finally, this paper concludes in Section 6.

2. Problem statement

Consider a linear stochastic time-varying system:

$$\begin{aligned} x_{k+1} &= A_k x_k + G_k d_k + \omega_k \\ y_k &= C_k x_k + v_k, \end{aligned} \quad (1)$$

where $x_k \in R^n$ is the state vector, $d_k \in R^m$ is the input vector, and $y_k \in R^p$ is the measurement vector at each time step k with $p \geq m$ and $n \geq m$. The process noise $\omega_k \in R^n$ and the measurement noise $v_k \in R^p$ are assumed to be mutually uncorrelated with zero-mean and a known covariance matrix, $Q_k = E[\omega_k \omega_k^T] \geq 0$ and $R_k = E[v_k v_k^T] > 0$, respectively. A_k , G_k and C_k are known matrices. Without loss of generality, we follow Gillijns and De Moor (2007) and Kitanidis (1987), and assume that G_k has a full column-rank. The initial state x_0 is independent of ω_k and v_k with a known mean \hat{x}_0 and covariance matrix $P_0 > 0$.

We consider the scenario where the input vector d_k is not fully observed at the level of interest but rather it is available only at an aggregate level. Specifically, let D_k be a $q_k \times m$ known matrix with $0 \leq q_k \leq m$ and F_{0k} an orthogonal complement of D_k^T such that $D_k F_{0k} = O_{q_k \times (m-q_k)}$ and $F_{0k}^T F_{0k} = I_{m-q_k}$, where O and I represent the zero matrix and identity matrix of appropriate dimensions. We suppose that the input data is available only on some linear combinations:

$$r_k = D_k d_k, \quad (2)$$

where r_k is available at each time step k . D_k is assumed to have a full row-rank; otherwise the redundant rows can be removed.

As pointed out in Li (2013), the matrix D_k characterizes the availability of input information at each time step k . It includes two extreme scenarios that are usually considered: (a) $q_k = m$ and D_k is an identity matrix, i.e. the complete input information is available; this is case that the classical Kalman filter can be applied; (b) $q_k = 0$, i.e. no information on the input variables is available; this is the problem investigated in Darouach and Zasadzinski (1997), Gillijns and De Moor (2007), Hsieh (2000) and Kitanidis (1987).

Throughout this paper, we use $\lambda(B)$ to denote any eigenvalue of a square matrix B . For any two symmetric matrices A and B with suitable dimensions, the notation $A \geq B$ is used if and only if

$A - B$ is non-negative definite. In addition, we use G_k^\perp to denote an orthogonal complement of G_k and $\Omega_k = [G_k, G_k^\perp]$. Define

$$\Pi_k = \begin{pmatrix} D_{k-1} \\ C_k G_{k-1} \end{pmatrix}. \quad (3)$$

3. Existence condition

To establish the existence condition of a general linear filter for system (1) and (2), we first consider an invertible linear transformation.

3.1. Transformation

Consider the following invertible matrix:

$$M_k = \begin{bmatrix} D_k & O_{q_k \times (n-m)} \\ O_{(n-m) \times m} & I_{n-m} \\ F_{0k}^T & O_{(m-q_k) \times (n-m)} \end{bmatrix} \Omega_k^{-1}.$$

It is straightforward to verify that $M_k G_k d_k$ can be expressed as:

$$\begin{aligned} M_k G_k d_k &= [D_k^T, O_{m \times (n-m)}, F_{0k}]^T d_k \\ &= [(D_k d_k)^T, (O_{(n-m) \times m} d_k)^T, (F_{0k}^T d_k)^T]^T \\ &= [r_k^T, O_{1 \times (n-m)}, (F_{0k}^T d_k)^T]^T \\ &= \tilde{r}_k + \tilde{G}_k \delta_k, \end{aligned} \quad (4)$$

where $\tilde{r}_k = [r_k^T, O_{1 \times (n-m)}, O_{1 \times (m-q_k)}]^T$, $\delta_k = F_{0k}^T d_k$ and $\tilde{G}_k = [O_{(m-q_k) \times q_k}, O_{(m-q_k) \times (n-m)}, I_{m-q_k}]^T$. We note that \tilde{r}_k is completely available due to Eq. (2).

Left-multiplying both sides of Eq. (4) by M_k^{-1} , $G_k d_k$ can be decoupled into two parts:

$$G_k d_k = M_k^{-1} \tilde{r}_k + M_k^{-1} \tilde{G}_k \delta_k. \quad (5)$$

From Eq. (5), the dynamics of x_{k+1} can be rewritten as:

$$\begin{aligned} x_{k+1} &= A_k x_k + M_k^{-1} \tilde{r}_k + M_k^{-1} \tilde{G}_k \delta_k + \omega_k \\ &= A_k x_k + u_k + F_k \delta_k + \omega_k, \end{aligned}$$

where $u_k = M_k^{-1} \tilde{r}_k$ is a known term, and F_k is given by

$$F_k = M_k^{-1} \tilde{G}_k = [G_k, G_k^\perp] \begin{bmatrix} F_{0k} \\ O \end{bmatrix} = G_k F_{0k}. \quad (6)$$

Consequently, linear system (1) with the partially observed inputs $r_k = D_k d_k$ can be equivalently represented by the following system:

$$\begin{aligned} x_{k+1} &= A_k x_k + u_k + F_k \delta_k + \omega_k \\ y_k &= C_k x_k + v_k. \end{aligned} \quad (7)$$

The above manipulation shows that a linear stochastic system with partially observed inputs (2) is equivalent to a linear system with unknown inputs; similar property is also found for linear descriptor systems (Hsieh, 2013).

3.2. Existence condition

In this subsection, we will establish the existence condition of a general, asymptotically stable and unbiased linear filter for system (7) and hence for its equivalent system, Eqs. (1) and (2).

Motivated by the linear filter structure in the literature (e.g. Darouach et al., 1994), we consider a general linear filter for discrete-time linear system (7) of the form

$$\hat{x}_{k+1} = E_k \hat{x}_k + J_k u_k + K_{k+1} y_{k+1}, \quad (8)$$

where the gain matrices E_k , J_k and K_{k+1} are to be designed. Based on (7) and (8), one can obtain the error dynamics $e_{k+1} = x_{k+1} - \hat{x}_{k+1}$:

$$\begin{aligned} e_{k+1} &= (A_k x_k + u_k + F_k \delta_k + \omega_k) - (E_k \hat{x}_k + J_k u_k + K_{k+1} y_{k+1}) \\ &= E_k e_k - (J_k - I + K_{k+1} C_{k+1}) u_k \\ &\quad + (A_k - K_{k+1} C_{k+1} A_k - E_k) x_k - (K_{k+1} C_{k+1} F_k - F_k) \delta_k \\ &\quad + (I - K_{k+1} C_{k+1}) \omega_k - K_{k+1} v_{k+1}. \end{aligned}$$

To ensure the filter is unbiased, it is required that the filtering error is independent of u_k , x_k and δ_k . In addition, it is expected that the error approaches to zero as time k increases. Hence the existence condition for filter (8) is given by:

- (i) E_k is stable (i.e., any eigenvalue of E_k satisfies $|\lambda(E_k)| < 1$);
- (ii) $E_k = A_k - K_{k+1} C_{k+1} A_k$;
- (iii) $K_{k+1} C_{k+1} F_k = F_k$;
- (iv) $J_k = I - K_{k+1} C_{k+1}$.

For system (1)–(2), however, the existence condition for system (7) should be expressed in terms of matrices A_k , G_k , C_k and D_k . For this end, we first state a lemma.

Lemma 1. For system (1)–(2), we have

$$\begin{aligned} \text{rank} \left(\begin{bmatrix} zI_n - A_k & -G_k \\ C_{k+1} & O \\ O & D_k \end{bmatrix} \right) \\ = \text{rank} \left(\begin{bmatrix} zI_n - A_k & -F_k \\ C_{k+1} & O \end{bmatrix} \right) + \text{rank}(D_k D_k^T). \end{aligned}$$

See the Appendix for proof. We now provide a condition for the existence of a general linear filter for a dynamic system with partially observed inputs.

Theorem 1. Suppose that both matrices D_k^T and G_k have a full column-rank. Then a sufficient condition for the existence of a general linear filter (8) for system (1)–(2) is given by:

$$\text{rank}(\Pi_{k+1}) = m \quad (9)$$

and for all $z \in \mathbb{C}$ (\mathbb{C} is the field of complex numbers) such that $|z| \geq 1$:

$$\text{rank} \left(\begin{bmatrix} zI_n - A_k & -G_k \\ C_{k+1} & O \\ O & D_k \end{bmatrix} \right) = n + m. \quad (10)$$

Proof. We note that we can select matrices $E_k = A_k - K_{k+1} C_{k+1} A_k$ and $J_k = I - K_{k+1} C_{k+1}$ to ensure that condition parts (ii) and (iv) are satisfied. Hence, we will focus on condition parts (i) and (iii). We first show that Eq. (9) guarantees there exists a matrix K_{k+1} such that condition part (iii) holds. We note

$$\begin{aligned} \begin{bmatrix} D_k \\ C_{k+1} G_k \end{bmatrix} \begin{bmatrix} F_{0k} & D_k^T \end{bmatrix} \\ = \begin{bmatrix} O_{q_k \times (m-q_k)} & D_k D_k^T \\ C_{k+1} G_k F_{0k} & C_{k+1} G_k D_k^T \end{bmatrix}. \end{aligned} \quad (11)$$

Since $[F_{0k}, D_k^T]$ is invertible and Π_{k+1} has a full column-rank, we obtain that $C_{k+1} G_k F_{0k} = C_{k+1} F_k$ (see Eq. (6)) is also of full column-rank, i.e.

$$\text{rank}(C_{k+1} F_k) = m - q_k. \quad (12)$$

Eq. (12) guarantees there exists a matrix K_{k+1} such that condition part (iii) holds.

Next, since $C_{k+1} F_k$ has a full column-rank, there exists an invertible matrix $N_k \in \mathbb{R}^{p \times p}$ such that

$$N_k C_{k+1} F_k = \begin{bmatrix} O_{(p-m+q_k) \times (m-q_k)} \\ I_{m-q_k} \end{bmatrix}.$$

The general solution K_{k+1} of $K_{k+1} C_{k+1} F_k = F_k$ is given by $K_{k+1} = [\Gamma_k, F_k] N_k$, where Γ_k can be any matrix of suitable dimensions and is to be designed for the gain matrix K_{k+1} .

Now define S_{1k} and S_{2k} such that

$$\begin{bmatrix} S_{1k} \\ S_{2k} \end{bmatrix} = N_k C_{k+1} A_k. \quad (13)$$

Then from condition part (ii), we can obtain

$$\begin{aligned} E_k &= A_k - K_{k+1} C_{k+1} A_k \\ &= A_k - [\Gamma_k, F_k] N_k C_{k+1} A_k = A_k - [\Gamma_k, F_k] \begin{bmatrix} S_{1k} \\ S_{2k} \end{bmatrix} \\ &= A_k - F_k S_{2k} - \Gamma_k S_{1k}. \end{aligned} \quad (14)$$

According to Anderson and Moore (1979, p. 342), the existence condition part (i) holds if and only if the following equivalent conditions hold:

- (a) $A_k - F_k S_{2k} - \Gamma_k S_{1k}$ is stable for a matrix Γ_k ;
- (b) $S_{1k} \eta = 0$ and $(A_k - F_k S_{2k}) \eta = \lambda \eta$ for some constant λ and vector η implies $|\lambda| < 1$ or $\eta = 0$.

The condition (b) can be expressed in the following equivalent form for all $z \in \mathbb{C}$ and $|z| \geq 1$:

$$\text{rank} \left(\begin{bmatrix} zI_n - A_k + F_k S_{2k} \\ S_{1k} \end{bmatrix} \right) = n. \quad (15)$$

The following identity, in conjunction with Lemma 1, shows that Eq. (15) is satisfied:

$$\begin{aligned} \text{rank} \left(\begin{bmatrix} zI_n - A_k & -F_k \\ C_{k+1} & O \end{bmatrix} \right) \\ = \text{rank} \left(\begin{bmatrix} I_n & O \\ -C_{k+1} & zI \end{bmatrix} \begin{bmatrix} zI_n - A_k & -F_k \\ C_{k+1} & O \end{bmatrix} \right) \\ = \text{rank} \left(\begin{bmatrix} zI_n - A_k & -F_k \\ C_{k+1} A_k & C_{k+1} F_k \end{bmatrix} \right) \\ = \text{rank} \left(\begin{bmatrix} I_n & O \\ O & N_k \end{bmatrix} \begin{bmatrix} zI_n - A_k & -F_k \\ C_{k+1} A_k & C_{k+1} F_k \end{bmatrix} \right) \\ = \text{rank} \left(\begin{bmatrix} zI_n - A_k & -F_k \\ S_{1k} & O \\ S_{2k} & I_{m-q_k} \end{bmatrix} \right) \\ = \text{rank} \left(\begin{bmatrix} zI_n - A_k + F_k S_{2k} & -F_k \\ S_{1k} & O \\ O & I_{m-q_k} \end{bmatrix} \right) \\ = \text{rank} \left(\begin{bmatrix} zI_n - A_k + F_k S_{2k} & O \\ S_{1k} & O \\ O & I_{m-q_k} \end{bmatrix} \right) \\ = \text{rank} \left(\begin{bmatrix} zI_n - A_k + F_k S_{2k} \\ S_{1k} \end{bmatrix} \right) + m - q_k. \end{aligned}$$

Hence, Eqs. (9) and (10) guarantee there exists a gain K_{k+1} such that: (a) $K_{k+1} C_{k+1} F_k = F_k$; and (b) $E_k = A_k - K_{k+1} C_{k+1} A_k$ is stable. This completes the proof.

Remarks. (i) Eq. (9) is the estimability condition for the filter developed in Li (2013) for system (1) with partially observed inputs (2). From the proof of Theorem 1, it also guarantees the unbiasedness of a general linear filter. In addition, Theorem 1 shows that to ensure the estimation error of a general linear filter is stable as time k increases, a detectability condition (10) needs to be met.

(ii) When condition parts (i)–(iv) are satisfied, the general linear filter (8) is given by

$$\hat{x}_{k+1} = (A_k - K_{k+1}C_{k+1}A_k)\hat{x}_k + (I - K_{k+1}C_{k+1})u_k + K_{k+1}y_{k+1}. \quad (16)$$

(iii) The error dynamics of the above filter (16) that satisfy condition (i)–(iv) become

$$e_{k+1} = (A_k - K_{k+1}C_{k+1}A_k)e_k + [I - K_{k+1}C_{k+1}, -K_{k+1}][\omega_k, v_{k+1}]^T. \quad (17)$$

3.3. Relationships with the existing filters

As mentioned earlier, system (1) with partially observed inputs (2) includes two important scenarios as its special cases: (a) the complete input information is available; and (b) no information on the input variables is available. In this subsection, we compare the developed existence condition in the previous subsection for partially observed inputs to the condition derived for the classical Kalman filter with complete information on the inputs, and to that of the filter with unknown inputs.

Theorem 2. The proposed existence condition for filter (8) in Theorem 1 reduces to: (a) the existence condition of the classical Kalman filter when the complete information on the inputs is available, i.e., D_k is invertible; and (b) the existence condition of the filter with unknown inputs, i.e. D_k is an empty matrix.

Proof. First, we consider the case that matrix D_k is invertible. It is clear that Eq. (9) is satisfied due to the non-singularity of D_k . In addition, we have

$$\begin{aligned} \text{rank} \left(\begin{bmatrix} zI_n - A_k & -G_k \\ C_{k+1} & O \\ O & D_k \end{bmatrix} \right) &= \text{rank} \left(\begin{bmatrix} zI_n - A_k & O \\ C_{k+1} & O \\ O & D_k \end{bmatrix} \right) \\ &= \text{rank} \left(\begin{bmatrix} zI_n - A_k \\ C_{k+1} \end{bmatrix} \right) + \text{rank}(D_k D_k^T). \end{aligned}$$

Since $\text{rank}(D_k D_k^T) = m$, the existence condition (10) reduces to

$$\text{rank} \left(\begin{bmatrix} zI_n - A_k \\ C_{k+1} \end{bmatrix} \right) = n, \quad \forall z \in \mathbb{C}, |z| \geq 1$$

which is the detectability condition of the classical Kalman filter (see, e.g. Anderson & Moore, 1979; Simon, 2006).

Next, we turn to the scenario where no information on the inputs d_k is available. Since matrix D_k reduces to a zero-by-zero empty matrix in this case, Eq. (9) becomes

$$\text{rank}(C_{k+1}G_k) = m. \quad (18)$$

In addition, Eq. (10) reduces to

$$\text{rank} \left(\begin{bmatrix} zI_n - A_k & G_k \\ C_{k+1} & O \end{bmatrix} \right) = n + m, \quad \forall z \in \mathbb{C}, |z| \geq 1. \quad (19)$$

Eqs. (18)–(19) are identical to the results for the filter with unknown inputs (Darouach & Zasadzinski, 1997). This completes the proof.

Theorem 2 shows that the obtained existence condition in this paper is a more generic condition. In addition, comparing the existence condition (9) and (10) of the general linear filter (8) for systems with partially available inputs to the existence condition (18)–(19), it can be seen that partial information on the unknown inputs has relaxed the existence condition of a general linear filter. In other words, with the information on the unknown inputs at an

aggregate level (2), it is more likely that the general linear filter (8) exists.

4. The filter with partially observed inputs

Now we focus on the filter proposed in Li (2013) for linear stochastic systems when the inputs are partially observed. Note that this filter was derived under the Bayesian framework with the assumption that ω_k and v_k follow a Gaussian distribution, and δ_k has a noninformative prior distribution. We summarize the results of the filter below. Define

$$\tilde{D}_k = \begin{pmatrix} D_k & O_{q_k \times (n-m)} \\ O_{(n-m) \times m} & I_{n-m} \end{pmatrix}.$$

Let $\tilde{M}_k = \tilde{D}_k \Omega_k^{-1}$. It is shown in Li (2013) that for system (1) with the input data available at an aggregate level (2), if matrix Π_k has a full column-rank, then the posterior distribution for x_k at any time step k is a Gaussian distribution with posterior mean $\hat{x}_{k|k}$ and posterior covariance matrix $P_{k|k}$ given by:

$$\begin{aligned} \hat{x}_{k|k} &= A_{k-1}\hat{x}_{k-1|k-1} + P_{k|k}\tilde{M}_{k-1}^T(\tilde{M}_{k-1}P_{k|k-1}\tilde{M}_{k-1}^T)^{-1} \\ &\quad \times \bar{r}_{k-1} + K_k(y_k - C_k A_{k-1}\hat{x}_{k-1|k-1}), \end{aligned} \quad (20)$$

and

$$\begin{aligned} P_{k|k} &= P_{k|k-1} - P_{k|k-1}C_k^T H_k^{-1} C_k P_{k|k-1} + [F_{k-1} \\ &\quad - P_{k|k-1}C_k^T H_k^{-1} C_k F_{k-1}][F_{k-1}^T C_k^T H_k^{-1} C_k F_{k-1}]^{-1} \\ &\quad \times [F_{k-1} - P_{k|k-1}C_k^T H_k^{-1} C_k F_{k-1}]^T, \end{aligned} \quad (21)$$

with

$$\begin{aligned} K_k &= P_{k|k-1}C_k^T H_k^{-1} + [F_{k-1} - P_{k|k-1}C_k^T H_k^{-1} C_k F_{k-1}] \\ &\quad \times [F_{k-1}^T C_k^T H_k^{-1} C_k F_{k-1}]^{-1} F_{k-1}^T C_k^T H_k^{-1}, \end{aligned} \quad (22)$$

$\bar{r}_k = [r_k^T, O^T]^T$, $P_{k|k-1} = A_{k-1}P_{k-1|k-1}A_{k-1}^T + Q_{k-1}$ and $H_k = C_k P_{k|k-1}C_k^T + R_k > 0$. Note that Eq. (9) guarantees Eq. (12) holds, and hence $F_{k-1}^T C_k^T H_k^{-1} C_k F_{k-1}$ is invertible in the above equations.

Under the Bayesian framework, $\hat{x}_{k|k}$ was shown to be a minimum mean square error (MMSE) estimate in Li (2013). However, no further properties of the filter were explored.

We now derive the dynamics of the state estimation error $e_k = x_k - \hat{x}_{k|k}$.

Lemma 2. The estimation error $e_k = x_k - \hat{x}_{k|k}$ of the filter (20)–(22) follows the recursive equation

$$e_k = (A_{k-1} - K_k C_k A_{k-1})e_{k-1} + [I - K_k C_k, -K_k][\omega_{k-1}, v_k]^T, \quad (23)$$

where K_k is given by Eqs. (21)–(22).

Proof. Let $W_{k-1} = P_{k|k}\tilde{M}_{k-1}^T(\tilde{M}_{k-1}P_{k|k-1}\tilde{M}_{k-1}^T)^{-1}$. The error dynamics of the filter (20)–(22) are given by

$$\begin{aligned} e_k &= A_{k-1}x_{k-1} + G_{k-1}d_{k-1} + \omega_{k-1} - A_{k-1}\hat{x}_{k-1|k-1} \\ &\quad - W_{k-1}\bar{r}_{k-1} - K_k(y_k - C_k A_{k-1}\hat{x}_{k-1|k-1}) \\ &= (A_{k-1} - K_k C_k A_{k-1})e_{k-1} + (G_{k-1} - K_k C_k G_{k-1})d_{k-1} \\ &\quad - W_{k-1}\bar{r}_{k-1} + (I - K_k C_k)\omega_{k-1} - K_k v_k. \end{aligned}$$

Noting that $\bar{r}_{k-1} = \tilde{M}_{k-1}G_{k-1}d_{k-1}$, we obtain

$$\begin{aligned} (G_{k-1} - K_k C_k G_{k-1})d_{k-1} - W_{k-1}\bar{r}_{k-1} \\ = [I - K_k C_k - W_{k-1}\tilde{M}_{k-1}]G_{k-1}d_{k-1}. \end{aligned} \quad (24)$$

Inserting (21) and (22) into (24), we can obtain (23) by noting that $I - K_k C_k - W_{k-1} \tilde{M}_{k-1} = O$. This completes the proof.

Lemma 2 shows that, for the gain K_k given in Eqs. (21)–(22), if $A_{k-1} - K_k C_k A_{k-1}$ is stable, the error of the developed filter in Li (2013) will be stable as time k increases. In addition, the estimation error Eq. (23) shares the same structure as that of Eq. (17), upon which we can conclude that the filter (20)–(22) falls into the filter family with the generic linear structure Eq. (8).

We now consider the error covariance matrix $P_{k|k}$.

Theorem 3. Let $\tilde{P}_{k|k}$ denote the error covariance matrix of any filter $\hat{x}_k(Y_k)$ based on the sequence of measurements $Y_k = \{y_0, y_1, \dots, y_k\}$. Then for linear system (1) with partially observed inputs (2), we have $\tilde{P}_{k|k} \geq P_{k|k}$, where $P_{k|k}$ is given by Eq. (21).

Proof. By definition, the conditional covariance matrix of the estimate $\hat{x}_k(Y_k)$ for given Y_k is

$$\tilde{P}_{k|k} = E\{[x_k - \hat{x}_k(Y_k)][x_k - \hat{x}_k(Y_k)]^T | Y_k\}.$$

It is easy to verify the following identity:

$$\begin{aligned} \tilde{P}_{k|k} &= E\{[x_k - \hat{x}_{k|k} + \hat{x}_{k|k} - \hat{x}_k(Y_k)] \\ &\quad \times [x_k - \hat{x}_{k|k} + \hat{x}_{k|k} - \hat{x}_k(Y_k)]^T | Y_k\} \\ &= P_{k|k} + E\{[\hat{x}_{k|k} - \hat{x}_k(Y_k)][\hat{x}_{k|k} - \hat{x}_k(Y_k)]^T | Y_k\} \\ &\quad + E\{[x_k - \hat{x}_{k|k}][\hat{x}_{k|k} - \hat{x}_k(Y_k)]^T | Y_k\} \\ &\quad + E\{[\hat{x}_{k|k} - \hat{x}_k(Y_k)][x_k - \hat{x}_{k|k}]^T | Y_k\}. \end{aligned}$$

Li (2013) shows that the estimated state vector $\hat{x}_{k|k}$ in Eq. (20) is the posterior mean conditional on the sequence of measurements $Y_k = \{y_0, y_1, \dots, y_k\}$. Hence, we have $E\{x_k | Y_k\} = \hat{x}_{k|k}$ and the last two terms on the right-hand side of the above equation vanish, i.e.

$$\tilde{P}_{k|k} = P_{k|k} + E\{[\hat{x}_{k|k} - \hat{x}_k(Y_k)][\hat{x}_{k|k} - \hat{x}_k(Y_k)]^T | Y_k\}.$$

We thus conclude that $\tilde{P}_{k|k}$ attains the minimum if and only if the second term of the right-hand side is equal to zero, i.e. $\hat{x}_k(Y_k) = \hat{x}_{k|k}$. This completes the proof.

Theorem 3 shows that the filter given by Eqs. (20)–(22) is optimal in the sense of both MMSE and minimum covariance matrix. This result is not only important in its own right but also useful in the subsequent asymptotic stability analysis.

5. Asymptotic stability

In this section, we discuss the asymptotic stability of the filter developed in Li (2013) for time-invariant system (1) and (2). We hence suppress the subscript k of matrices A_k , G_k , C_k , D_k , Q_k and R_k .

We note from Lemma 2 that the covariance matrix in Eq. (21) can be re-written as

$$\begin{aligned} P_{k|k} &= (A - K_k C A) P_{k-1|k-1} (A - K_k C A)^T \\ &\quad + (I - K_k C) Q (I - K_k C)^T + K_k R K_k^T. \end{aligned} \quad (25)$$

Under the condition given in Theorem 1 and in conjunction with Theorem 3 that the covariance matrix of the filter given by Eqs. (20)–(22) is optimal, it can be shown that the covariance matrix $P_{k|k}$ in Eq. (25) is bounded for all k and for an arbitrary bounded initial covariance $P_{0|0}$. On the basis of boundedness of $P_{k|k}$ and inspired by the approaches in Anderson and Moore (1979) and Fang and Callafon (2012), we can further show the following result. The proof is omitted here for lack of space and is available upon request.

Theorem 4. If the condition in Theorem 1 is satisfied and $(A, Q^{\frac{1}{2}})$ is stabilizable, then the covariance matrix $P_{k|k}$ of the filter (20)–(22) will converge to a unique fixed positive semi-definite matrix \bar{P} for any given initial condition $P_{0|0}$. Moreover, with the associated limiting gain matrices \bar{K} , the time-invariant filter is also stable, i.e. all the eigenvalues of $A - \bar{K} C A$ satisfy $|\lambda(A - \bar{K} C A)| < 1$.

It is of interest to compare the asymptotic stability condition obtained with partially observed inputs to the asymptotic stability conditions when the complete information on the inputs is available and when the inputs are completely unknown. This is investigated in the following theorem. It shows that Theorem 4 provides a unified approach to accommodating asymptotic stability conditions in a variety of filtering scenarios.

Theorem 5. The asymptotic stability condition for the filter (20)–(22) in Theorem 4 reduces to: (a) the asymptotic stability condition of the classical Kalman filter when the complete information on the inputs is available, i.e. D is invertible; and (b) the asymptotic stability condition of the filter with unknown inputs, i.e. D is an empty matrix.

Proof. First, we note that when matrix D is invertible, the asymptotic stability condition reduces to: (a) (A, C) is detectable; and (b) $(A, Q^{\frac{1}{2}})$ is stabilizable. These are the asymptotic stability conditions of the classical Kalman filter (see, e.g. Anderson & Moore, 1979).

Next, when no information on the inputs is available, we know from Theorem 2 that Eq. (10) in Theorem 1 reduces to Eq. (19). In addition, condition $(A, Q^{\frac{1}{2}})$ along with $R > 0$ (and hence $R^{\frac{1}{2}} > 0$) can guarantee that the matrix below has a full row-rank, i.e.,

$$\text{rank} \left(\begin{bmatrix} A - e^{jw} I & G & Q^{\frac{1}{2}} & O \\ e^{jw} C & O & O & R^{\frac{1}{2}} \end{bmatrix} \right) = n + p, \quad \forall w \in [0, 2\pi]. \quad (26)$$

Eqs. (19) and (26) are identical to the asymptotic stability condition for the filter with unknown inputs (Darouach & Zasadzinski, 1997). This completes the proof.

6. Conclusions

This paper has established existence and asymptotic stability conditions for the recently developed filter with partially observed inputs in Li (2013). The obtained existence and asymptotic stability conditions provide a unified approach to accommodating a variety of filtering scenarios as its special cases, including the important Kalman filtering and the unknown input filtering problems. In practice, information on inputs and/or outputs may sometimes be only partially available in applications. This work takes a further step towards the development of more generic filtering techniques where different levels of information are exploited.

Acknowledgements

We thank the associate editor and reviewers for their very helpful comments on the earlier versions of this paper.

Appendix. Proof of Lemma 1

$$\begin{aligned} \text{rank} \left(\begin{bmatrix} zI_n - A_k & -G_k \\ C_{k+1} & O \\ O & D_k \end{bmatrix} \right) \\ = \text{rank} \left(\begin{bmatrix} zI_n - A_k & -G_k \\ C_{k+1} & O \\ O & D_k \end{bmatrix} \begin{bmatrix} I_n & O \\ O & [F_{0k} D_k^T] \end{bmatrix} \right) \end{aligned}$$

$$\begin{aligned}
&= \text{rank} \left(\begin{bmatrix} zI_n - A_k & -G_k F_{0k} & -G_k D_k^T \\ C_{k+1} & O & O \\ O & D_k F_{0k} & D_k D_k^T \end{bmatrix} \right) \\
&= \text{rank} \left(\begin{bmatrix} zI_n - A_k & -F_k \\ C_{k+1} & O \end{bmatrix} \right) + \text{rank}(D_k D_k^T).
\end{aligned}$$

References

- Anderson, B. D., & Moore, J. B. (1979). *Optimal filtering*. Englewood Cliffs: Prentice-Hall.
- Bejarano, F. J., Floquet, T., Perruquetti, W., & Zheng, G. (2013). Observability and detectability of singular linear systems with unknown inputs. *Automatica*, 49(3), 793–800.
- Cheng, Y., Ye, H., Wang, Y. Q., & Zhou, D. H. (2009). Unbiased minimum-variance state estimation for linear systems with unknown input. *Automatica*, 45(2), 485–491.
- Corless, M., & Tu, J. (1998). State and input estimation for a class of uncertain systems. *Automatica*, 34(6), 757–764.
- Darouach, M., & Zasadzinski, M. (1997). Unbiased minimum variance estimation for systems with unknown exogenous inputs. *Automatica*, 33(4), 717–719.
- Darouach, M., Zasadzinski, M., & Xu, S. J. (1994). Full-order observers for linear systems with unknown inputs. *IEEE Transactions on Automatic Control*, 39(3), 606–609.
- Fang, H.-Z., & Callafon, R. A. D. (2012). On the asymptotic stability of minimum-variance unbiased input and state estimation. *Automatica*, 48(12), 3183–3186.
- Gillijns, S., & De Moor, B. (2007). Unbiased minimum-variance input and state estimation for linear discrete-time systems. *Automatica*, 43(1), 111–116.
- Hou, M., & Muller, P. C. (1992). Design of observers for linear systems with unknown inputs. *IEEE Transactions on Automatic Control*, 37(6), 871–875.
- Hsieh, C. S. (2000). Robust two-stage Kalman filters for systems with unknown inputs. *IEEE Transactions on Automatic Control*, 45(2), 2374–2378.
- Hsieh, C. S. (2010). On the global optimality of unbiased minimum-variance state estimation for systems with unknown inputs. *Automatica*, 46(4), 708–715.
- Hsieh, C. S. (2011). Optimal filtering for systems with unknown inputs via the descriptor Kalman filtering method. *Automatica*, 47(10), 2313–2318.
- Hsieh, C. S. (2013). State estimation for descriptor systems via the unknown input filtering method. *Automatica*, 49(5), 1281–1286.
- Kitanidis, P. K. (1987). Unbiased minimum-variance linear state estimation. *Automatica*, 23(6), 775–778.
- Li, B. (2009). A non-Gaussian Kalman filter with application to the estimation of vehicular speed. *Technometrics*, 51(2), 162–172.
- Li, B. (2013). State estimation with partially observed inputs: a unified Kalman filtering approach. *Automatica*, 49(3), 816–820.
- Liang, Y., Chen, T.-W., & Pan, Q. (2010). Optimal linear state estimator with multiple packet dropouts. *IEEE Transactions on Automatic Control*, 55(6), 1428–1443.
- Simon, D. (2006). *Optimal state estimation: Kalman, H_∞ , and non-linear approaches*. New York: Wiley.



Jinya Su received his B.Sc. degree in the School of Mathematics and Statistics from Shandong University, Weihai, China in 2011. He is currently a Ph.D. candidate in the Department of Aeronautical and Automotive Engineering, Loughborough University, Loughborough, UK. His research interests include disturbance observer design and its applications in fault diagnosis, disturbance rejection control, etc.



Baibing Li received a B.Sc. degree from Yunnan University, Kunming, China, an M.Sc. degree from Shanghai Jiao Tong University, Shanghai, China, and an M.Sc. degree from Vrije Universiteit Brussel, Brussels, Belgium. In 1991, he received a Ph.D. degree from the Management School, Shanghai Jiao Tong University. He was a Postdoctoral Research Fellow with Katholieke Universiteit Leuven, Leuven, Belgium, and a Research Associate with Newcastle University, Newcastle upon Tyne, UK. In 2001, he was appointed as a Lecturer at Newcastle University. In 2004, he moved to the School of Business and Economics, Loughborough University, Loughborough, UK, as a Lecturer, where he was subsequently appointed as a Reader in 2007 and a Professor in 2011. His current research interests cover Bayesian statistical modelling and forecasting for Gaussian and non-Gaussian dynamic problems in various management areas. In recent years, much of his work has also involved transport and traffic management such as transportation demand analysis, travel behaviour modelling, and intelligent transportation systems.

He is a member of IEEE and a member of the Royal Statistical Society.



Wen-Hua Chen received the M.Sc. and Ph.D. degrees from Northeast University, Shenyang China, in 1989 and 1991, respectively. From 1991 to 1996, he was a Lecturer and then Associate Professor with the Department of Automatic Control, Nanjing University of Aeronautics and Astronautics, Nanjing, China. From 1997 to 2000, he held a research position and then a Lecturer in control engineering with the Centre for Systems and Control, University of Glasgow, Glasgow, UK. In 2000, he moved to the Department of Aeronautical and Automotive Engineering, Loughborough University, Loughborough, UK, as a Lecturer, where he was appointed as a Professor in 2012. His research interests include the development of advanced control strategies (Nonlinear Model Predictive Control, Disturbance Observer Based Control, etc.) and their applications in aerospace engineering. Currently, much of his work has also involved in the development of Unmanned Autonomous Intelligent Systems.

He is a Senior Member of IEEE and a Fellow of IET.